DIFFERENCE EQUATION OF THE COLORED JONES POLYNOMIAL FOR TORUS KNOT

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A . We prove that the N-colored Jones polynomial for the torus knot $\mathcal{T}_{s,t}$ satisfies the second order difference equation, which reduces to the first order difference equation for a case of $\mathcal{T}_{2,2m+1}$. We show that the A-polynomial of the torus knot can be derived from the difference equation. Also constructed is a q-hypergeometric type expression of the colored Jones polynomial for $\mathcal{T}_{2,2m+1}$.

1. I

The N-colored Jones polynomial $J_{\mathcal{K}}(N)$ is one of the quantum invariants for knot $\mathcal{K}$. It is associated with the N-dimensional irreducible representation of $sl(2)$, and is powerful to classify knots. Motivated by "volume conjecture" [12, 18] saying that a hyperbolic volume of the knot complement dominates an asymptotic behavior of the colored Jones polynomial $J_{\mathcal{K}}(N)$ at $q = e^{2\pi i/N}$, it receives much interests toward a geometrical and topological interpretation of the quantum invariants.

Recently another intriguing structure of the colored Jones polynomial was put forward; it was shown that the N-colored Jones polynomial $J_{\mathcal{K}}(N)$ can be written in a q-hypergeometric form, and that it satisfies a recursion relation with respect to N [6]. It was further demonstrated for trefoil and figure-eight knot [5] that a recursion relation is related to the A-polynomial $A_{\mathcal{K}}(L, M)$ (see also Ref. 4), which denotes an algebraic curve of eigenvalues of the $SL(2, \mathbb{C})$ representation of the boundary torus of knot $\mathcal{K}$ [3]. As the A-polynomial contains many geometrical informations such as the boundary slopes of the knot, this "AJ conjecture" may help our geometrical understanding of the colored Jones polynomial.

In this article, we study torus knot $\mathcal{T}_{s,t}$ where s and t are relatively prime integers. We prove that the N-colored Jones polynomial $J_{\mathcal{K}}(N)$ for the torus knots $\mathcal{K} = \mathcal{T}_{s,t}$ satisfies the second order recursion relation (7) [Theorem 4], which reduces to the first order (9) only in a case of $\mathcal{K} = \mathcal{T}_{2,2m+1}$ [Theorem 6]. Furthermore we shall show that this difference operator gives the A-polynomial of the torus knot as was demonstrated in Ref. 5 [Theorem 7]. Throughout this article, we normalize the colored Jones polynomial to be

$$J_{\text{unknot}}(N) = 1$$

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2. C J P

The Alexander polynomial $\Delta_{\mathcal{K}}(A)$ for the torus knot $\mathcal{K} = \mathcal{T}_{s,t}$ is known to be (see *e.g.* Ref. 14)

$$\Delta_{\mathcal{T}_{s,t}}(A) = \frac{(A^{1/2} - A^{-1/2})(A^{st/2} - A^{-st/2})}{(A^{s/2} - A^{-s/2})(A^{t/2} - A^{-t/2})} \tag{1}$$

We see that an inverse of the Alexander polynomial is expanded in $A \to \infty$ as

$$\frac{A^{1/2} - A^{-1/2}}{\Delta_{\mathcal{T}_{s,t}}(A)} = \sum_{n=0}^{\infty} \chi_{2st}(n)\, A^{-n/2} \tag{2}$$

where $\chi_{2st}(n)$ is the periodic function with modulus $2\,s\,t$ [11];

$n \mod 2\,s\,t$	$s\,t-s-t$	$s\,t-t+s$	$s\,t+t-s$	$s\,t+s+t$	others
$\chi_{2st}(n)$	1	-1	-1	1	0

Using this periodic function, we define the function $K_{s,t}(N)$ by

$$K_{s,t}(N) = q^{\frac{1}{4}N\left(2(st-s-t)-stN\right)} \sum_{k=0}^{stN} \chi_{2st}(s\,t\,N-k)\, q^{\frac{k^2-(st-s-t)^2}{4st}} \tag{3}$$

Theorem 1. *The N-colored Jones polynomial for the torus knot $\mathcal{T}_{s,t}$ is given by*

$$J_{\mathcal{T}_{s,t}}(N) = \frac{q^{\frac{1}{2}(s-1)(t-1)(1-N)}}{1-q^{-N}}\, K_{s,t}(N) \tag{4}$$

As we have defined the function $K_{s,t}(N)$ from an expansion of the Alexander polynomial, this theorem reveals a connection between the colored Jones polynomial and the Alexander polynomial for a case of the torus knot. It should be noted that a relationship between these polynomials is known for arbitrary knot $\mathcal{K}$ based on a slightly different expansion as the Melvin–Morton conjecture [16, 19], which was proved in Ref. 2.

To prove Theorem 1, we use a previously known result for the colored Jones polynomial.

Proposition 2 ([17]). *The N-colored Jones polynomial for the torus knot $\mathcal{T}_{s,t}$ is computed as*

$$J_{\mathcal{T}_{s,t}}(N) = \frac{q^{\frac{1}{4}st(1-N^2)}}{q^{\frac{N}{2}} - q^{-\frac{N}{2}}} \sum_{r=-\frac{N-1}{2}}^{\frac{N-1}{2}} \left(q^{str^2-(s+t)r+\frac{1}{2}} - q^{str^2-(s-t)r-\frac{1}{2}}\right) \tag{5}$$

Proof of Theorem 1. We first prove for a case of N being even. We have

$$\sum_{k=0}^{stN} \chi_{2st}(s\,t\,N-k)\,q^{\frac{k^2-(st-s-t)^2}{4st}} = \sum_{k=0}^{stN} \chi_{2st}(k)\,q^{\frac{k^2-(st-s-t)^2}{4st}}$$
$$= \sum_{k=0}^{\frac{N}{2}-1} \left(q^{k(stk+st-s-t)} - q^{(sk+s-1)(tk+1)} - q^{(sk+1)(tk+t-1)} + q^{(k+1)(stk+s-1)}\right)$$
$$= \sum_{k=-\frac{N}{2}}^{\frac{N}{2}-1} \left(q^{k(stk+st-s-t)} - q^{(sk+s-1)(tk+1)}\right)$$

In the second equality, we have substituted a non-zero value of the periodic function $\chi_{2st}(k)$. In the last equality, we have replaced k with $-k-1$ in both the third and the fourth terms in a parenthesis. By further replacing k with $k-\frac{1}{2}$ in the last expression and comparing with an explicit form given in eq. (5), we get an assertion of the theorem.

For a case of N being odd, we can prove it in a same manner. □

Proposition 3. *Let the function $K_{s,t}(N)$ be defined by eq.* (3)*. Then it satisfies the following difference equation;*

$$K_{s,t}(N) = 1 - q^{s(1-N)-1} - q^{t(1-N)-1} + q^{(s+t)(1-N)} + q^{st(2-N)-s-t}\,K_{s,t}(N-2) \quad (6)$$

Proof. We decompose a sum in eq. (3) into $\sum_{k=0}^{st(N-2)} + \sum_{k=st(N-2)}^{stN}$. From the first sum we obtain $K_{s,t}(N-2)$. The second sum can be written explicitly as what appeared in eq. (6) using a property of $\chi_{2st}(k)$. □

As we have obtained a relationship between the colored Jones polynomial $J_{\mathcal{T}_{s,t}}(N)$ and $K_{s,t}(N)$, it is straightforward to obtain a following theorem.

Theorem 4. *The N-colored Jones polynomial for the torus knot $\mathcal{T}_{s,t}$ fulfills a recursion relation of the second order;*

$$J_{\mathcal{T}_{s,t}}(N) = \frac{q^{\frac{1}{2}(s-1)(t-1)(1-N)}}{1-q^{-N}}\left(1 - q^{s(1-N)-1} - q^{t(1-N)-1} + q^{(s+t)(1-N)}\right) + \frac{1-q^{2-N}}{1-q^{-N}}\,q^{st(1-N)-1}\,J_{\mathcal{T}_{s,t}}(N-2) \quad (7)$$

We should note that this difference equation can be directly derived by use of Morton's expression (5). A benefit of our expression (4) is in reducing the difference equation (7) into the first order difference equation in a case of $\mathcal{T}_{2,2m+1}$.

Proposition 5 (see *e.g.* Ref. 9)**.** *The function $K_{2,2m+1}(N)$ satisfies the difference equation of the first order,*

$$K_{2,2m+1}(N) = 1 - q^{1-2N} - q^{2m(1-N)-N}\,K_{2,2m+1}(N-1) \quad (8)$$

Proof. We note that the function $\chi_{8m+4}(n)$ has an anti-periodicity, $\chi_{8m+4}(n+4m+2) = -\chi_{8m+4}(n)$. Then a proof follows in a same method with that of eq. (6). □

This proposition simplifies the difference equation of the colored Jones polynomial $J_{\mathcal{K}}(N)$ for $\mathcal{K} = \mathcal{T}_{2,2m+1}$ as follows;

Theorem 6. *The N-colored Jones polynomial for the torus knot $\mathcal{T}_{2,2m+1}$ solves the difference equation of the first order;*

$$J_{\mathcal{T}_{2,2m+1}}(N) = q^{m(1-N)} \frac{1-q^{1-2N}}{1-q^{-N}} - q^{m-(2m+1)N} \frac{1-q^{1-N}}{1-q^{-N}} J_{\mathcal{T}_{2,2m+1}}(N-1) \tag{9}$$

This recursion relation coincides with a result in Ref. 5 for the trefoil $m = 1$ (we need to replace q with q^{-1}). We remark that in Ref. 7 by a different approach proposed was the difference equation of the colored Jones polynomial for the torus knot $\mathcal{T}_{2,2m+1}$, which is much involved.

3. A-P

In Ref. 5 the "AJ conjecture" is proposed; the *homogeneous* difference equation of the colored Jones polynomial $J_{\mathcal{K}}(N)$ for knot $\mathcal{K}$ gives the A-polynomial $A_{\mathcal{K}}(L, M)$ for knot $\mathcal{K}$. More precisely we rewrite the difference equation of the colored Jones polynomial, $\sum_{k\geq 0} a_k\, J_{\mathcal{K}}(N + k) = 0$, into a form, $\mathcal{A}_{\mathcal{K}}(E, Q; q)\, J_{\mathcal{K}}(N) \equiv \sum_{k\geq 0} a_k(Q, q)\, E^k\, J_{\mathcal{K}}(N) = 0$, where the operators Q and E act on $J_{\mathcal{K}}(N)$ as

$$E\, J_{\mathcal{K}}(N) = J_{\mathcal{K}}(N + 1)$$

$$Q\, J_{\mathcal{K}}(N) = q^N\, J_{\mathcal{K}}(N)$$

Then a claim of Ref. 5 is that the A-polynomial $A_{\mathcal{K}}(L, M)$ coincides with

$$A_{\mathcal{K}}(L, M) = \mathcal{A}_{\mathcal{K}}(L, M^2; q = 1) \tag{10}$$

This AJ conjecture was checked with a help of `Mathematica` package for trefoil, figure-eight knots [5], and for 5_2, 6_1 knots [21].

For our case of the torus knot $\mathcal{T}_{s,t}$, we can easily check that applying above procedure the difference equations (7) and (9) reproduce the A-polynomial for the torus knots given in Ref. 23 (see also Refs. 3, 20);

$$A_{\mathcal{T}_{s,t}}(L, M) = \begin{cases} (L-1)\,(-1 + L^2\, M^{2st}) & \text{for } s, t > 2 \\ (L-1)\,(1 + L\, M^{2(2m+1)}) & \text{for } (s, t) = (2, 2m+1) \end{cases}$$

We can thus conclude that

Theorem 7. *AJ conjecture* (10) *proposed in Ref. 5 is true for the torus knots $\mathcal{K} = \mathcal{T}_{s,t}$.*

4. *q*-H F C J P

At the end of this article, we comment on an explicit form of the colored Jones polynomial for the torus knot. Recalling a result of Ref. 9, we see that the colored Jones polynomial for the torus knot $\mathcal{T}_{2,2m+1}$ can be written in a form of the q-hypergeometric function.

Hereafter we use a standard notation of the q-product and the q-binomial coefficient (see *e.g.* Ref. 1);

$$(x)_n = (x;q)_n = \prod_{k=1}^{n} (1 - x\, q^{k-1})$$

$$\begin{bmatrix} n \\ m \end{bmatrix}_q = \frac{(q)_n}{(q)_{n-m}\,(q)_m}$$

Theorem 8 ([9]). *Let the function $H_{2,2m+1}(x)$ be defined by*

$$H_{2,2m+1}(x) = \sum_{k_m \geq \cdots \geq k_2 \geq k_1 \geq 0}^{\infty} (x)_{k_m+1}\, x^{k_m} \left(\prod_{i=1}^{m-1} q^{k_i(k_i+1)}\, x^{2k_i} \begin{bmatrix} k_{i+1} \\ k_i \end{bmatrix}_q \right) \tag{11}$$

for $|q| < 1$ and $|x| < 1$. Then we have

$$H_{2,2m+1}(x) = \sum_{n=0}^{\infty} \chi_{8m+4}(n)\, q^{\frac{n^2-(2m-1)^2}{8(2m+1)}}\, x^{\frac{n-(2m-1)}{2}} \tag{12}$$

and it satisfies

$$H_{2,2m+1}(x) = 1 - q\, x^2 - q^{2m}\, x^{2m+1}\, H_{2,2m+1}(q\, x) \tag{13}$$

For $m = 1$ case, see also Ref. 1 [Chap. 2, p. 29] and Ref. 22.

With an expression (11), we can take a limit $x \to q^{-N}$; an infinite sum terminates into a finite sum due to $(q^{-N})_k = 0$ for $k > N$. Comparing eq. (8) with eq. (13) we obtain the colored Jones polynomial for the torus knot $\mathcal{T}_{2,2m+1}$ as follows;

Proposition 9. *The N-colored Jones polynomial for the torus knot $\mathcal{T}_{2,2\,m+1}$ is given by*

$$J_{\mathcal{T}_{2,2m+1}}(N) = q^{m(1-N)} \sum_{k_m \geq \cdots \geq k_2 \geq k_1 \geq 0}^{\infty} (q^{1-N})_{k_m}\, q^{-N k_m} \left(\prod_{i=1}^{m-1} q^{k_i(k_i+1-2N)} \begin{bmatrix} k_{i+1} \\ k_i \end{bmatrix}_q \right) \tag{14}$$

We see that the colored Jones polynomial for the trefoil $m = 1$ coincides with results in Refs. 8, 13. It should be remarked that in Ref. 15 constructed was the colored Jones polynomial for the twist knot, which gives a different expression of the colored Jones polynomial for the trefoil. By applying a twisting formula given in Ref. 15, we have another expression of the colored Jones polynomial in the form of *cyclotomic expansion* in a sense of Ref. 8 as follows;

Proposition 10. *The N-colored Jones polynomial for the torus knot $\mathcal{T}_{2,2m+1}$ is written as*

$$J_{\mathcal{T}_{2,2m+1}}(N) = q^{m(1-N^2)} \sum_{k_m \geq \cdots \geq k_2 \geq k_1 \geq 0}^{\infty} \frac{(q^{1-N})_{k_m} (q^{1+N})_{k_m}}{(q)_{k_m}} \times \left(\prod_{i=1}^{m-1} q^{(k_i - k_m)(k_i - k_m - 1)} \begin{bmatrix} k_{i+1} \\ k_i \end{bmatrix}_q \right) \tag{15}$$

For the torus knot $\mathcal{T}_{s,t}$ with $s, t > 2$, we do not know a general expression of the colored Jones polynomial in terms of the q-hypergeometric function; based on Refs. 10, 11, we only have

$$J_{\mathcal{T}_{3,4}}(N) = q^{3(1-N)} \sum_{n=0}^{\infty} (q^{1-N})_n \, q^{-2Nn} \times \left(\sum_{k=0}^{\lfloor (n-1)/2 \rfloor} q^{2k(k+1-N)+N} \begin{bmatrix} n \\ 2k+1 \end{bmatrix}_q + \sum_{k=0}^{\lfloor n/2 \rfloor} q^{2k(k+1-N)} \begin{bmatrix} n+1 \\ 2k+1 \end{bmatrix}_q \right) \tag{16}$$

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